APPROACH

- Algorithm design technique for solving optimization problems
- Start with a feasible solution
- Repeat the following step until no improvement can be found:
  - change the current feasible solution to a feasible solution with a better value of the objective function
- Return the last feasible solution as optimal
- Note: Typically, a change in a current solution is "small" (local search)
- Major difficulty: Local optimum vs. global optimum

EXAMPLE: SIMPLEX METHOD

Linear Programming (LP) Problem:

- optimize a linear function of several variables subject to linear constraints:
- maximize (or minimize) $c_1 x_1 + \ldots + c_n x_n$
- subject to $a_{i\,1}x_1 + \ldots + a_{in} x_n \leq (\text{or} \geq \text{or} =)\ b_i\,,\ i = 1,\ldots,m$
  $x_1 \geq 0, \ldots , x_n \geq 0$
- The function $z = c_1 x_1 + \ldots + c_n x_n$ is called the *objective function*;
- constraints $x_1 \geq 0, \ldots , x_n \geq 0$ are called *nonnegativity constraints*

Possible Outcomes

1. Problem has a finite optimal solution, which may not be unique
2. Problem could be unbounded: the objective function of maximization (minimization) LP problem is unbounded from above (below) on its feasible region
3. Problem could be infeasible: there are no points satisfying all the constraints, i.e. the constraints are contradictory

Extreme Point Theorem:

Any LP problem with a nonempty bounded feasible region has an optimal solution; moreover, an optimal solution can always be found at an *extreme point* of the problem's feasible region

Example 1 - Final Optimal Solution

maximize $3x + 5y$ subject to

$$x + y \le 4$$
$$x + 3y \le 6$$
$$x \ge 0, \quad y \ge 0$$

The *Feasible region* is the set of points defined by the constraints

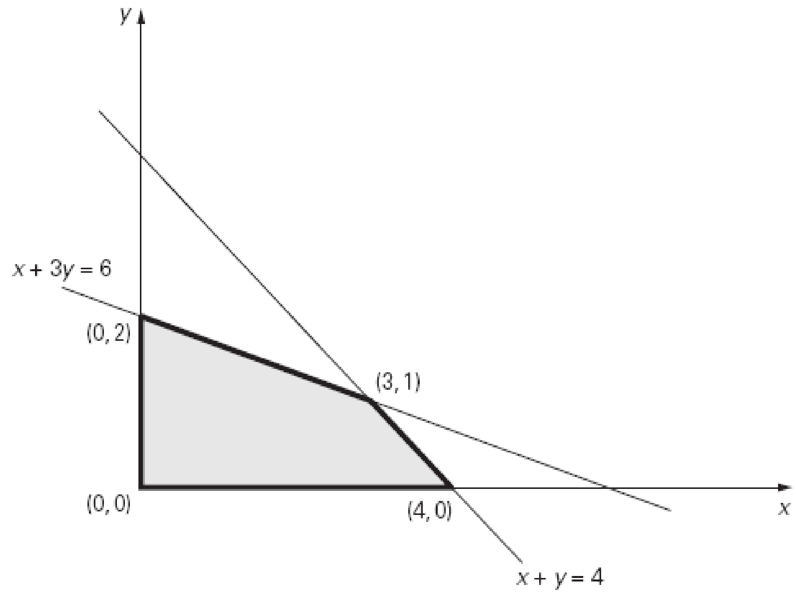


FIGURE 10.1 Feasible region of problem (10.2).

An *Optimal Solution* to the LP Problem is a point for which the value of the objective function is maximized.

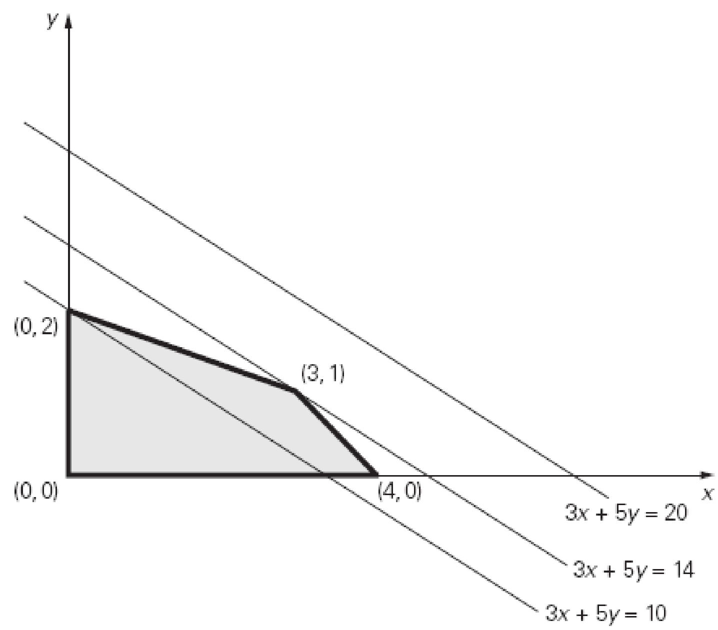


FIGURE 10.2 Solving a two-dimensional linear programming problem geometrically.

Example 2 - Unbounded problem

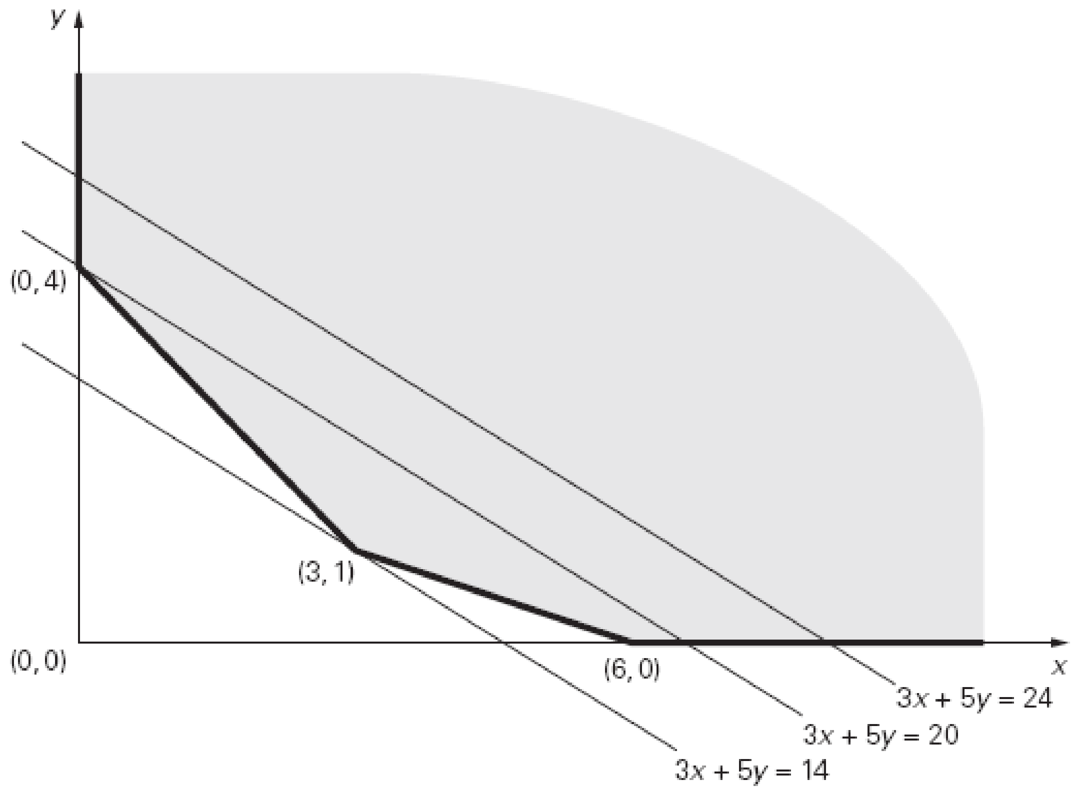


FIGURE 10.3 Unbounded feasible region of a linear programming problem with constraints $x + y \geq 4$, $x + 3y \geq 6$, $x \geq 0$, $y \geq 0$, and three level lines of the function $3x + 5y$.

Example 3 - Unfeasible problem

maximize $3x + 5y$
subject to $x + y \geq 4$
$x + y \leq 2$
$x \geq 0, \; y \geq 0$

Simplex Method

- The classic method for solving LP **maximization** problems; one of the most important algorithms ever invented
- Invented by George Dantzig in 1947
- Based on the iterative improvement idea:
- Generates a sequence of adjacent points of the problem's feasible region with improving values of the objective function until no further improvement is possible

Step 0: Initialization

Step 0.1: convert inequalities

$$\begin{array}{ll} \text{maximize} & 3x + 5y \\ \text{subject to} & x + y \le 4 \\ & x + 3y \le 6 \\ & x \ge 0,\ y \ge 0 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \text{maximize} & 3x + 5y + 0u + 0v \\ \text{subject to} & x + y + u = 4 \\ & x + 3y + v = 6 \\ & x \ge 0,\ y \ge 0,\ u \ge 0,\ v \ge 0 \end{array}$$

Variables u and v, transforming inequality constraints into equality constrains, are called *slack variables*

Step 0.2: calculate basic feasible solution

- A *basic solution* to a system of m linear equations in n unknowns (n ≥ m) is obtained by setting n – m variables to 0 and solving the resulting system to get the values of the other m variables. The variables set to 0 are called *nonbasic*; the variables obtained by solving the system are called *basic*.
- A basic solution is called *feasible* if all its (basic) variables are nonnegative.
- Example $x + y + u = 4$

  $x + 3y + v = 6$

  (0, 0, 4, 6) is basic feasible solution

  (x, y are non basic; u, v are basic)
- There is a 1-1 correspondence between extreme points of LP's feasible region and its basic feasible solutions.
- Calculate value of function at that solution:

  $3x + 5y + 0u + 0v = 0$

<ins>Simplex Tableau representation of step 0.2</ins>

		x	y	u	v		
Basic	u	1	1	1	0	4	
variables	v	1	3	0	1	6	value of z at
Objective row →		-3	-5	0	0	0	← (0,0,4,6)
		↑ Reverse coefficients					

<ins>Simplex Algorithm</ins>

- **Step 0 [Initialization]**

  Present a given LP problem in standard form and set up initial tableau.
- **Step 1 [Optimality test]**
  - If all entries in the objective row are nonnegative — stop: the tableau represents an optimal solution.
- **Step 2 [Find entering variable]**
  - Select (the most) negative entry in the objective row.

  Mark its column to indicate the entering variable and the pivot column.
- **Step 3 [Find departing variable]**
  - For each positive entry in the pivot column, calculate the θ-ratio by dividing that row's entry in the rightmost column by its entry in the pivot column.
  - (If there are no positive entries in the pivot column — stop: the problem is unbounded.)
  - Find the row with the smallest θ-ratio, mark this row to indicate the departing variable and the pivot row.
- **Step 4 [Form the next tableau]**
  - Divide all the entries in the pivot row by its entry in the pivot column.
  - Subtract from each of the other rows, including the objective row, the new pivot row multiplied by the entry in the pivot column of the row in question.
  - Replace the label of the pivot row by the variable's name of the pivot column and go back to Step 1.

<u>Example</u>

Simplex Tableau	**Basic feasible solution**	**z**
Tableau 1	(0, 0, 4, 6)	0
Tableau 2	(0, 2, 2, 0)	10
Tableau 3	(3, 1, 0, 0)	14

Tableau 1:

	x	y	u	v	
u	1	1	1	0	4
$\leftarrow$ v	1	3	0	1	6
	−3	−5	0	0	0
		$\uparrow$			

Tableau 2:

	x	y	u	v	
$\leftarrow$ u	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	2
y	$\frac{1}{3}$	1	0	$\frac{1}{3}$	2
	$-\frac{4}{3}$	0	0	$\frac{5}{3}$	10
	$\uparrow$				

Tableau 3:

	x	y	u	v	
x	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	3
y	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	1
	0	0	2	1	14

<u>Notes on the Simplex Method</u>

- Finding an initial basic feasible solution may pose a problem
- Theoretical possibility of cycling
- Typical number of iterations is between m and 3m, where m is the number of equality constraints in the standard form
- Worse-case efficiency is exponential

Example #2

Maximise

$x+y$

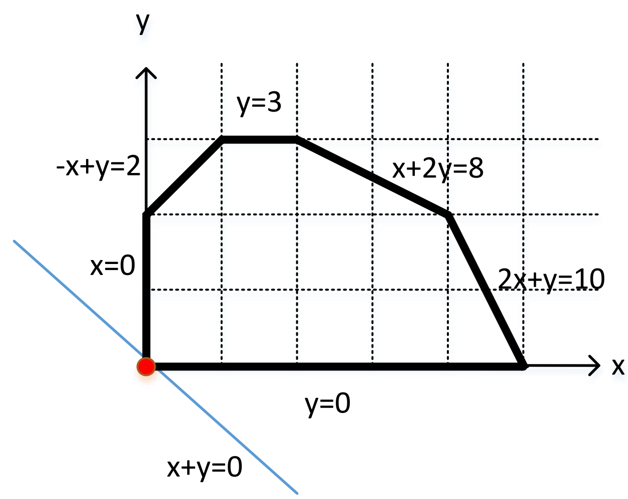
For

$y \leq 3$

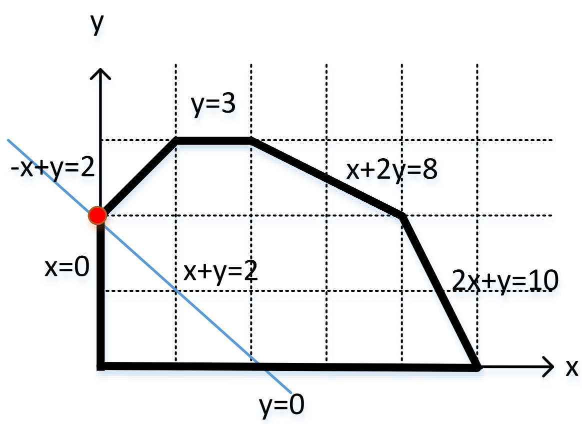
$-x+y \leq 2$

$x+2y \leq 8$

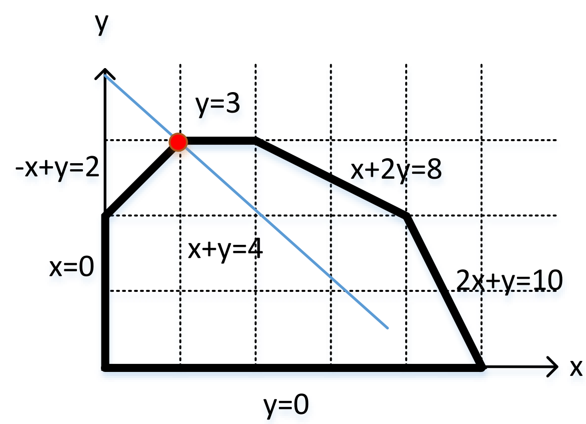
$2x+y \leq 10$



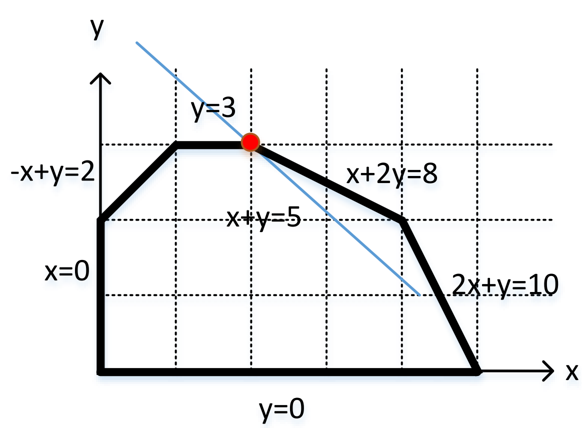
1



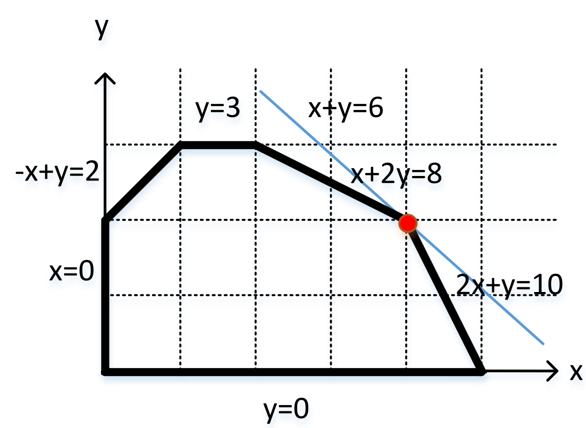
2



3



4



5

		x	**y**	**a**	**b**	**c**	**d**			
		0	0	3	2	8	10		**θ ratio**	
y ≤ 3	a	0	1	1	0	0	0	3	3	a = 3-y
-x+y ≤ 2	b	-1	1	0	1	0	0	2	2	b = 2+x-y
x+2y=8	c	1	2	0	0	1	0	8	4	c = 8-x-2y
2x+y ≤ 10	d	2	1	0	0	0	1	10	10	d = 10-2x-y
MAX-x-y = 0		-1	-1	0	0	0	0	0		MAX=x+y

		x	**y**	**a**	**b**	**c**	**d**			
		0	2	1	0	4	8		**θ ratio**	
y ≤ 3	a	1	0	1	-1	0	0	1	1	a = 3-y
-x+y ≤ 2	y	-1	1	0	1	0	0	2	-2	b = 2+x-y
x+2y=8	c	3	0	0	-2	1	0	4	4/3	c = 8-x-2y
2x+y ≤ 10	d	3	0	0	-1	0	1	8	8/3	d = 10-2x-y
MAX-x-y = 0		-2	0	0	1	0	0	2		MAX=x+y

		x	**y**	**a**	**b**	**c**	**d**			
		1	3	0	0	1	5		**θ ratio**	
y ≤ 3	x	1	0	1	-1	0	0	1	-1	a = 3-y
-x+y ≤ 2	y	0	1	1	0	0	0	3	undef	b = 2+x-y
x+2y=8	c	0	0	-3	1	1	0	1	1	c = 8-x-2y
2x+y ≤ 10	d	0	0	-3	2	0	1	5	5/2	d = 10-2x-y
MAX-x-y = 0		0	0	2	-1	0	0	4		MAX=x+y

		x	**y**	**a**	**b**	**c**	**d**			
		2	3	0	1	0	3		**θ ratio**	
y ≤ 3	x	1	0	-2	0	1	0	2	-1	a = 3-y
-x+y ≤ 2	y	0	1	1	0	0	0	3	3	b = 2+x-y
x+2y=8	b	0	0	-3	1	1	0	1	-1/3	c = 8-x-2y
2x+y ≤ 10	d	0	0	3	0	-2	1	3	1	d = 10-2x-y
MAX-x-y = 0		0	0	-1	0	1	0	5		MAX=x+y

		x	**y**	**a**	**b**	**c**	**d**			
		4	2	1	4	0	0		**θ ratio**	
y ≤ 3	x	1	0	0	0	-1/3	2/3	4	-12	a = 3-y
-x+y ≤ 2	y	0	1	0	0	-2/3	-1	2	-3	b = 2+x-y
x+2y=8	b	0	0	0	1	-1	1	4	-4	c = 8-x-2y
2x+y ≤ 10	a	0	0	1	0	-2/3	1/3	1	-3/2	d = 10-2x-y
MAX-x-y = 0		0	0	2	0	-1	1	5		MAX=x+y